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LETTER TO THE EDITOR

New family of solvable 1D Heisenberg models

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Abstract. Starting from a Calogero–Sutherland model with the hyperbolic interaction confined by an external field with Morse potential, we construct a Heisenberg spin chain with exchange interaction $\propto 1/\sinh^2 x$ on a lattice given in terms of the zeros of Laguerre polynomials. Varying the strength of the Morse potential, the Haldane–Shastry and harmonic spin chains are reproduced. The spectrum of the models in this class is found to be that of a classical one-dimensional Ising chain with non-uniform nearest-neighbour coupling in a non-uniform magnetic field which allows us to study the thermodynamics in the limit of infinite chains.

Studies of Heisenberg spin chains with exchange coupling proportional to certain functions decaying like the inverse square of the distance have unveiled surprisingly simple properties which allow for a detailed study of these many-body systems. Among these properties are exact wavefunctions of Jastrow product form and very simple spectra. Most notably, the Haldane–Shastry model [1, 2] with trigonometric ($\propto 1/\sin^2 x$) exchange admits an interpretation as the generalization of the concept of an 'ideal gas' to the case of fractional statistics [3–5].

All of these models can be related to Calogero–Sutherland models of particles with an internal degree of freedom (spin) using the exchange-operator formalism [6]. Within this approach, the spin chains are obtained by 'freezing out' the kinematic degrees of freedom in these models, thereby giving a pure exchange model of particles interacting with an inverse-square potential on a lattice given by the static-equilibrium positions of the Calogero model. Within this formalism, it has been possible to construct a family of operators commuting with the Hamiltonian and among themselves for the Haldane–Shastry model [7] and a model with rational exchange related to the $1/x^2$ Calogero model confined by a harmonic potential well [8–10] (due to its spectrum this model will be referred to as the 'harmonic spin chain' in the following). Although it has not been possible so far to obtain the Hamiltonian from this construction directly, this is to be considered as the proof of integrability of these models.

In the limit of infinite chains, the above considerations have been applied to the hyperbolic chain with exchange $\propto 1/\sinh^2 x$ on a translationally-invariant lattice. A generalization of this case to *finite non-uniform* lattices which reduces to the Haldane–Shastry and the harmonic spin chain in certain limits is presented in this letter. We start with the Hamiltonian of an N-particle Calogero–Sutherland system of particles interacting

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with an external field with Morse potential given by

$$\mathcal{H}_{\mathrm{CS}} = \sum_{j}^{N} \left[\frac{p_j^2}{2} + 2\tau^2 (\mathrm{e}^{4x_j} - 2\mathrm{e}^{2x_j}) \right] + \sum_{j<k}^{N} \frac{1}{\sinh^2(x_j - x_k)}. \tag{1}$$

It contains a real parameter $\tau > 0$ related to the strength of the Morse potential. The integrability of the dynamical model defined by (1) has been proven in [12] for arbitrary τ. Below, we shall consider this model allowing an additional exchange interaction of the particles now carrying an internal degree of freedom (spin). Upon appropriate rescaling of the dynamical variables, the kinematic degrees of freedom in such a model can be eliminated yielding a spin lattice with sites defined by the classical equilibrium positions of the particles [7, 8]. These are given in terms of the set of nonlinear algebraic equations

$$-\sum_{k\neq j}^{N} \frac{z_k(z_j + z_k)}{(z_j - z_k)^3} + \tau^2 (z_j - 1) = 0 \qquad (j = 1, \ldots, N) \tag{2}$$

where we have introduced $z_j = \mathrm{e}^{2x_j}$. A general solution to (2) is unknown. However, following the observation by Calogero [11] for the $1/x^2$ case, we assume that the $\{z_j\}$ are the roots of some polynomial $p_N(z) = \prod_{j=1}^{N}(z - z_j)$ obeying the second-order differential equation [12]

$$Ap_N'' + Bp_N' + \lambda_N p_N = 0 \tag{3}$$

where A and B are polynomials in z and λ_N is some constant. Choosing $\lambda_N = 2N\tau$, we find that (3) reads

$$y\frac{\mathrm{d}^2 p_N(y)}{\mathrm{d}y^2} + (-y + 1 + \Gamma)\frac{\mathrm{d}p_N(y)}{\mathrm{d}y} + Np_N(y) = 0 \qquad y = 2\tau z \tag{4}$$

with $\Gamma = 2(\tau - N) + 1$ from which we identify p_N as the Laguerre polynomial $L_N^{(\Gamma)}(2\tau z)$. Let us note the following properties of $L_N^{(\Gamma)}$.

(i) For $\Gamma > -1$, all roots of $L_N^{(\Gamma)}$ are real positive numbers.

(ii) $L_N^{(\Gamma)}$ has multiple zeros if and only if $\Gamma = -n$, $n = 2, 3, \ldots, N$. At these values of Γ, one can write

$$L_N^{(-n)}(y) = \frac{(N-n)!}{N!}(-1)^{(n)} y^n L_{N-n}^{(n)}(y).$$

(iii) As $\Gamma = -n + \varepsilon$, $\varepsilon \to 0$, the n roots of $L_N^{(\Gamma)}$ approaching 0 have the following asymptotic behaviour:

$$z_j \sim \mathrm{const}\, |\varepsilon|^{1/n} \exp\left(\frac{2\pi \mathrm{i} j}{n}\right) \qquad j = 1, \ldots, n. \tag{5}$$

Introducing variables $z_j = 1 + \tau^{-1/2}\zeta_j$ and taking the limit of $\tau \to \infty$, the rational Calogero model related to the harmonic spin chain is obtained. In this case, the lattice points ζ_j are the roots of the Hermite polynomial $H_N(\zeta)$. As can be seen from (5), for the special values of Γ being a negative integer $\geqslant -N$, the lattice separates into two parts not coupled by the interaction term in the Hamiltonian. Upon rescaling, one of them is given by the nth root of unity (5), which results in a lattice of equally-spaced sites with periodic boundary conditions. The corresponding spin chain is the Haldane–Shastry model with interaction $\propto 1/\sin^2 x$. The other part corresponds to the hyperbolic model with Γ replaced by $-\Gamma$. Finally, as $\Gamma \to -N$, the model reduces to the trigonometric model.

Following Fowler and Minahan [7], we now consider the model where N bosonic particles are sitting on different points z_j satisfying (2) and allow the exchange of particle positions as the only dynamical process. Denoting the corresponding Hermitian exchange operator for particles i and j by $\mathcal{M}_{ij}$, we choose the Hamiltonian of this system to be

$$\mathcal{H}_{\mathrm{ex}} = \sum_{j<k}^{N} h_{jk}\mathcal{M}_{jk} \qquad h_{jk} = \frac{1}{4\sinh^2(x_j - x_k)} = \frac{z_j z_k}{(z_j - z_k)^2}. \tag{6}$$

To prove integrability of the model, one needs to construct a family of symmetric operators commuting with $\mathcal{H}_{\mathrm{ex}}$. Using the properties

$$\mathcal{M}_{ij} z_i = z_j \mathcal{M}_{ij} \qquad \mathcal{M}_{ij} z_k = z_k \mathcal{M}_{ij} \qquad \text{for } i \neq k \neq j$$
$$\mathcal{M}_{ijk} = \mathcal{M}_{ik}\mathcal{M}_{ij} = \mathcal{M}_{ij}\mathcal{M}_{jk} = \mathcal{M}_{jk}\mathcal{M}_{ik}$$

of the exchange operators $\mathcal{M}_{ij}$, one finds that the operators

$$h_j = \tfrac{1}{4}(\hat{\pi}_j + \omega_j)(\hat{\pi}_j - \omega_j) - \tau^2(z_j - 1)^2 + \mathrm{const}\,\omega_j \tag{7}$$

where

$$\hat{\pi}_j = \sum_{k\neq j}^{N} \frac{z_j + z_k}{z_j - z_k}\mathcal{M}_{jk} \qquad \omega_j = \sum_{k\neq j}^{N} \mathcal{M}_{jk}$$

obey the relation $[h_j, \mathcal{H}_{\mathrm{ex}}] = 0$ if the coordinates of the points of our hyperbolic lattice satisfy the set of equilibrium equations (2). Note that at some value of the constant in (7), these operators can be factorized

$$h_j = a_j^+ a_j^- \qquad a_j^{\pm} = \tfrac{1}{2}\left(\hat{\pi}_j \pm \omega_j\right) \pm \delta\tau(z_j - 1) \qquad \delta^2 = 1. \tag{8}$$

Commutators of two operators from this set are a bit more complicated than in the case of the harmonic chain

$$[h_j, h_k] = 2\delta\tau(h_j\mathcal{M}_{jk} - \mathcal{M}_{jk}h_j) + \omega_j\mathcal{M}_{jk}h_j - h_j\mathcal{M}_{jk}\omega_j.$$

The symmetric combinations of $\{h_j\}$ commuting with the Hamiltonian (6) can now be written in the form

$$\mathcal{I}_m = \sum_{j=1}^{N} h_j^m. \tag{9}$$

For the Haldane–Shastry model [7] and the harmonic spin chain [8], it can be shown that operators similar to these $\mathcal{I}_m$ also commute among themselves. We have not been able to prove this in the general case considered here. After tedious calculation, we were only able to show that $[\mathcal{I}_n, \mathcal{I}_m] = 0$ for $1 \leqslant n, m \leqslant 3$.

As usual, the operators $\mathcal{M}_{jk}$ can be related to exchange operators acting not on the particle positions but on their internal degree of freedom as they are defined in the space of bosonic, i.e. totally symmetric wavefunctions. This leads us to consider the Hamiltonian

$$\mathcal{H} = -\sum_{j<k}^{N} h_{jk} \frac{\sigma_j \cdot \sigma_k - 1}{2} = C_N - \sum_{j<k}^{N} h_{jk} \mathcal{P}_{jk} \qquad C_N = \tfrac{1}{24} N(N-1)(3\Gamma + 2N - 1) \tag{10}$$

with the spin exchange operator $\mathcal{P}_{jk} = \frac{1}{2}(\sigma_j \cdot \sigma_k - 1)$. The σ_j^α are Pauli matrices acting on the jth site of the lattice. The same procedure gives a trivial exchange operator $\tilde{\mathcal{I}}_1 = -\frac{1}{3}\sum \mathcal{P}_{ijk}$ for the spin analogue of $\mathcal{I}_1$. The next operator in this sequence $\tilde{\mathcal{I}}_2$, already contains four spin exchange terms with non-uniform exchange couplings.

To construct eigenstates of the Hamiltonian (10), we start from the ferromagnetic vacuum $|0\rangle = |\uparrow\uparrow \cdots \uparrow\rangle$ and consider states with given magnetization

$$|\psi^{(M)}\rangle = \sum_{j_1 < \cdots < j_M}^{N} \psi(j_1 \ldots j_M) \prod_{s=1}^{M} \sigma_s^- |0\rangle. \tag{11}$$

Using the fact that the z_j are roots of the Nth Laguerre polynomial, we find that the *eigenstates in the one-magnon sector ($M = 1$) have amplitudes*

$$\psi_m(j) \propto z_j^m \frac{L_{N-m-1}^{(\Gamma+2m)}(2\tau z_j)}{L_{N-1}^{(\Gamma)}(2\tau z_j)} \qquad m = 0, \ldots, N-1. \tag{12}$$

Their energies $E_m^{(1)} = \epsilon_m$ are given by the following quadratic dispersion law:

$$\epsilon_m = \frac{m}{2}(\Gamma + m). \tag{13}$$

Next we have studied the two-magnon sector ($M = 2$). The amplitudes $\psi(j_1, j_2)$ can be written as polynomials in $\{z_j^{-1}\}$. No closed expression for these amplitudes has been found. Nevertheless, the eigenproblem can be solved analytically and we were able to find the complete set of $N(N-1)/2$ eigenvalues which can be written as

$$E_{m,n}^{(2)} = \epsilon_m + \epsilon_n(1 - \delta_{m,n-1}) \qquad 0 \leqslant m < n \leqslant N-1 \tag{14}$$

with the single-magnon energies ϵ_m given by (13).

Finally, we considered the M-magnon sector. Within the ansatz

$$\psi = \frac{\prod_{\lambda>\mu}^{M} (z_{j_\lambda} - z_{j_\mu})^2 F(z_{j_1} \ldots z_{j_M})}{\prod_{\nu=1}^{M} z_{j_\nu} p_N'(z_{j_\nu})} \tag{15}$$

where F is some symmetric polynomial in $\{z\}$, the corresponding part of the spectrum which comprises $\frac{(N-M+1)!}{M!(N-2M+1)!}$ eigenvalues is found to be additive

$$E_{\{m_k\}}^{(M)} = \sum_{k=1}^{M} \epsilon_{m_k} \tag{16}$$

with the dispersion (13), where the integers m_k obey the set of restrictions

$$m_k < m_{k+1} - 1 \qquad 0 \leqslant m_k \leqslant N-1. \tag{17}$$

While our solution of the M-particle sector is not complete, we make the following hypothesis concerning the spectrum of the class of models (10).

Hypothesis. All the eigenvalues of $\mathcal{H}$ can be written in compact form

$$E_{n_1 \dots n_N} = \sum_{k=1}^{N-1} \epsilon_k n_{k+1}(1 - n_k) \tag{18}$$

where $\epsilon_k = \frac{1}{2}k(\Gamma + k)$ and $\{n_k\} = 0, 1$. As a consequence, the operator $\mathcal{H} = -2\sum_{j<k}^{N} h_{jk}\sigma_j\sigma_k$ is equivalent, up to unitary transformation, to the Hamiltonian of the classical 1D Ising chain in a non-uniform magnetic field

$$\begin{aligned}\mathcal{H}_I &= \sum_{k=1}^{N-1} \epsilon_k(\sigma_{k+1} - \sigma_k - \sigma_{k+1}\sigma_k) \\ &= \epsilon_{N-1}\sigma_N + \sum_{k=0}^{N-2}[\sigma_{k+1}(\epsilon_k - \epsilon_{k+1}) - \sigma_{k+1}\sigma_{k+2}\varepsilon_{k+1}]\end{aligned}$$

with $\{\sigma_k\} = \pm 1$.

This hypothesis comprises all our analytical results from the two-magnon sector (14) (where our analytic proof is complete) as well as for the case of arbitrary $M > 2$ (16). Moreover, it is, in fact, motivated and supported by the picture of lattice separation at $-N \leqslant \Gamma = -n < -1$ (where $\epsilon_n = 0$) mentioned above and, for $\Gamma = -N$, reproduces the Ising Hamiltonian that follows for the Haldane–Shastry case in the appropriate limit of the Hubbard model with $1/r$ hopping [13]. We have also confirmed the Ising-like form of the spectrum in our general case by establishing the analytic correspondence in the limit $\Gamma \to \infty$ to the effective Hamiltonian of the harmonic chain (equation (20) in [9]). Finally, we have checked this hypothesis numerically to give the correct spectrum for lattices of lengths up to $N = 12$ at several choices of the parameter τ in equations (1) and (2).

The simple form of spectrum (18) allows us to compute the free energy in the thermodynamic limit by using the transfer-matrix method. The partition function on a finite lattice can be written as

$$\begin{aligned}\mathcal{Z}_N = \frac{1}{2}\,\mathrm{trace}\Bigg\{ &\begin{pmatrix} 1/w_1 & -1/w_1 \\ 1 & 1 \end{pmatrix} \prod_{k=1}^{N-2}\left[\begin{pmatrix} 1 + w_k & 0 \\ 0 & 1 - w_k \end{pmatrix}\mathcal{B}_k\right] \\ &\times \begin{pmatrix} (1 + w_{N-1})^2 & (1 + w_{N-1})^2 \\ (1 - w_{N-1})^2 & (1 - w_{N-1})^2 \end{pmatrix}\Bigg\}\end{aligned} \tag{19}$$

where

$$\mathcal{B}_k = \frac{1}{2}\begin{pmatrix} 1 + \dfrac{w_k}{w_{k+1}} & 1 - \dfrac{w_k}{w_{k+1}} \\ 1 - \dfrac{w_k}{w_{k+1}} & 1 + \dfrac{w_k}{w_{k+1}} \end{pmatrix} \qquad w_k = \mathrm{e}^{-\frac{1}{2}\beta\epsilon_k}. \tag{20}$$

To perform the thermodynamic limit $N \to \infty$, we rescale the magnon energies (13) with a factor $1/N^2$ and find the leading term of (19) to be $Z \sim \exp(\beta E_0)\prod_{k=1}^{N-2}(1 + w_k)$ with some proper E_0 renormalizing the ground-state energy to zero. From this expression, one obtains for the free energy per site

$$f = -\frac{E_0}{N} - \frac{1}{\beta}\int_0^1 \mathrm{d}x\, \ln(1 + \mathrm{e}^{-\beta\epsilon(x)/2}). \tag{21}$$

Using the quasiparticle dispersion $\epsilon(x) = x(\gamma + x)$ (the renormalization of the magnon energies leads to $\gamma = \Gamma/N$ when taking the thermodynamic limit of (13)), this can be written for $-1 < \gamma < 0$ as

$$f = -\frac{1}{\beta}\left(\int_0^{-\gamma} \mathrm{d}x \, \ln(1 + \mathrm{e}^{\beta\epsilon(x)/2}) + \int_{-\gamma}^{1} \mathrm{d}x \, \ln(1 + \mathrm{e}^{-\beta\epsilon(x)/2})\right). \qquad (22)$$

It is easy to see that for $\gamma = -1$ this result coincides with the free energy of the Haldane–Shastry model found in [3] (see also [14]). Note, however, that for our general model there is no restriction for the value of γ. It is also worth noting that changing the sign of the quasiparticle dispersion or, equivalently, the exchange in (10) from ferromagnetic to antiferromagnetic, changes the free energy by a temperature-independent term only (due to the different ground-state energy). This has been previously noticed in the special case of the Haldane–Shastry model ($\gamma = -1$) [3] and is a consequence of the possibility to describe these models in terms of an effective Ising model (18).

Since the total spin of the effective Ising model is proportional to the z-component of the total spin in the original model, one can also study the effect of an external magnetic field. For $\Gamma > -1$, the largest eigenvalue of (10) in the M-magnon sector is

$$E_{\max} = \tfrac{1}{6}M(3(N - M)(N - M + \Gamma) + M^2 - 1) \qquad \text{for } M \leqslant \tfrac{1}{2}N \qquad (23)$$

from which we conclude that the ground-state magnetization of the *anti*ferromagnetic chain in a magnetic field h is given by $\mathcal{M}(h) = \frac{1}{4}(\sqrt{\gamma^2 + 8h} - \gamma)$ for $h \leqslant h_s = \frac{1}{2}(\gamma + 1)$. Beyond h_s, the magnetization is saturated at $\frac{1}{2}$. For finite temperatures, the transfer-matrix method yields

$$\mathcal{M}(\beta, h) = \frac{1}{2}\int_0^1 \mathrm{d}x \, \frac{\sinh \beta h/2}{\sqrt{\exp(-\beta\epsilon(x)) + \sinh^2 \beta h/2}}. \qquad (24)$$

In this letter, we have constructed a new family of solvable Heisenberg spin chains. As in the previously known Haldane–Shastry and harmonic models, their spectra can be given in terms of an effective Ising Hamiltonian. This mapping allows us to study the thermodynamics of these models in detail. Generalization to other exchange-type models such as interacting $SU(N)$ spins or electrons subject to a supersymmetric t–J or Hubbard-type Hamiltonian are easily constructed by considering different representations of the permutation operators in (10). At the same time, a number of interesting questions, especially regarding the construction of a complete set of conserved quantities, remains open.

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